Quark Flavor Physics Review

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I review the status of lattice-QCD calculations relevant to quark flavor physics. The recent availability of physical-mass ensembles with large physical volumes generated by a growing number of lattice collaborations is an exciting development and I discuss their impact on the landscape of lattice flavor physics calculations. The activities of the newly formed FLAG-2 collaboration which aims to provide averages of quantities calculated in lattice QCD that are relevant for quark flavor physics are also discussed. My talk covers (a subset of) the same quantities reviewed by FLAG-2, including results for K, $D_{(s)}$, and $B_{(s)}$ meson decay constants and semileptonic decay form factors, as well as for hadronic matrix elements of neutral (K, D, and $B_{(s)}$) meson mixing. I also briefly discuss recent progress towards understanding nonleptonic K decay and long-distance contributions to Δm_K.

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1. Introduction and motivation

Quark flavor physics has a history that is rich in important discoveries that were crucial in the formulation and understanding of the Standard Model (SM) of particle physics. With the recent discovery at the LHC of a Higgs boson and with the non-observation of other new particles in the few hundred GeV to TeV energy range, quark flavor physics continues to be an important component of the intensity frontier and of the overall high energy physics program. The goal of the current (and planned) experimental intensity frontier effort is to constrain and possibly even discover beyond the SM (BSM) physics by confronting precision measurements with SM predictions. For many quark flavor physics processes, lattice QCD is a crucial ingredient for obtaining sufficiently precise SM predictions. Hence, the worldwide theoretical effort to calculate the needed hadronic matrix elements in lattice QCD is central to this program.

A relevant and very recent example that illustrates the interplay between experiment and theory in quark-flavor physics is the rare decay $B_s \to \mu^+ \mu^-$. This decay was first observed by the LHC*b* collaboration. Its branching fraction is now measured by LHC*b* and CMS with a combined accuracy of 24% [1] and we can expect the experimental accuracy to rapidly improve once the LHC turns on again. Recently, the Wilson coefficient for this process was calculated including three-loop perturbative QCD and two-loop EW corrections [2], greatly reducing the corresponding contributions to the error budget of the SM prediction. Lattice QCD and CKM inputs are the dominant sources of error and the resulting SM prediction has a total uncertainty of about 6% [2]. The lattice QCD input in this case is the B_s meson decay constant (which is taken with a 2% error) and enters into the SM prediction quadratically. I should also note that the CKM inputs depend on our knowledge of other hadronic matrix elements, calculated in lattice QCD, such as B_s mixing parameters and $B \to D^{(*)}$ form factors. An alternate SM prediction can be obtained by normalizing the branching fraction to the B_s mass difference, Δm_{B_s} [3]. This removes the CKM inputs and the square of the decay constant from the SM expression, but adds the B_s mixing bag parameter. Since the bag parameter is currently known from lattice QCD with a precision of about 5%, this alternate approach doesn't improve the accuracy of the SM prediction [3] at the moment. The $B_s \to \mu^+ \mu^-$ decay, being loop-induced in the SM, is a sensitive probe of new physics, and already provides strong constraints on BSM models at the current level of accuracy in both experiment and SM theory.

Worldwide, there are many different lattice groups (using different lattice methods) who calculate hadronic matrix elements relevant for a variety of weak decay processes of K, $D_{(s)}$, and $B_{(s)}$ mesons. With so many groups calculating the same matrix elements using different methods, but all providing phenomenologically relevant results with complete error budgets, it is desirable to combine all the lattice results for a given quantity into one number (and error bar). A few years ago, two such averaging efforts were started, the first by Laiho, Lunghi, and Van de Water (LLV) [5] provided averages for both heavy- and light-quark quantities, while the second, the Flavianet Lattice Averaging working group (FLAG-1) [6] focused on just light-quark quantities. Both groups computed averages from lattice results with complete error budgets only. However, FLAG-1 also provided guidance about the quality of the systematic error estimates in the form of quality criteria for the dominant sources of systematic error, including continuum extrapolation, chiral extrapolation, finite volume effects, and renormalization. These two efforts are now merged into one, aptly

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named the Flavor Lattice Averaging Group (FLAG-2), while at the same time increasing the membership to 28 people, in order to include researchers from as many of the big collaborations working in lattice flavor physics as possible. Like LLV, FLAG-2 now reviews heavy- and light-quark quantities, while much of the structure adopted by FLAG-1 is retained in FLAG-2. In particular, individual working groups of three FLAG-2 members are responsible for reviewing and averaging the lattice results for a particular quantity (or group of quantities). The quality criteria developed in FLAG-1 for the various sources of systematic error in lattice QCD calculations are also adopted in FLAG-2. In addition, new criteria for heavy-quark treatment and continuum extrapolation for the new heavy-quark quantities were developed. In FLAG only results based on simulations with $N_f \geq 2$ flavors of dynamical fermions are considered, and separate averages are provided for each value of $N_f = 2, 2+1, 2+1+1$. A preliminary version of the FLAG-2 review was posted on the FLAG website (http://itpwiki.unibe.ch/flag) in Summer 2013, with the full review posted in the archives in October [7]. The lattice averaging effort is meant to complement the work of the Particle Data Group (PDG) [8] and the Heavy Flavor Averaging Group (HFAG) [9] who provide averages of experimental measurements. Combining the lattice averages together with experimental averages allows us to obtain SM parameters and constraints on BSM theories with the best possible precision.

The quantities that I discuss in this review are covered by four (of the seven) FLAG-2 working groups. In particular, my review focuses on leptonic decay constants and semileptonic form factors for kaons, $D_{(s)}$ and $B_{(s)}$ mesons, and on neutral (K, D, and $B_{(s)}$) meson mixing matrix elements (of local four-fermion operators). I also briefly mention recent results relevant for nonleptonic kaon decay as well as the long-distance contribution to neutral kaon mixing, neither of which are part of the FLAG-2 effort. The FLAG-2 review includes lattice results made public before the end of April,[1] but here I also include relevant results presented after the end of April, at the Lattice conference, and through Fall 2013. For this purpose, the summary figures of Ref. [7], which show all the lattice calculations of a given quantity in comparison to the FLAG-2 averages, are modified to include these recent results for comparison purpose only. The FLAG-2 averages are unchanged, since these new results have not been evaluated yet. Nevertheless the new results give an impression of how the lattice-QCD calculations for a given quantity are evolving. The modified figures are shown in the physics sections below.

Before discussing the physics results in Sections 3-6 I present a short overview of heavy-quark methods, a discussion of the FLAG-2 criteria relevant for heavy-quark quantities, as well as a brief summary of the simulations in Section 2.

2. Heavy-quark methods, FLAG-2 criteria, and status of simulations

A detailed description of the various heavy-quark methods that are used in lattice-QCD calculations of heavy (charm and bottom) quark quantities is given in Appendix A.1.3 of Ref. [7]. Here I provide a brief overview.

Charm quarks can be treated with light-quark methods (provided that the action is at least $O(a)$ improved), since with currently available lattice spacings $am_c \gtrsim 0.15$. A number of different light-

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[1]An update of Ref. [7] is currently in preparation which will include results made public by the end of November 2013.

quark actions are now being used for charm-quark quantities (see Section 4 for more details). I note that an interesting new method is introduced in Ref. [10]. Starting with the so-called Brillouin fermion action [11], a tree-level $O(a^2)$ improved version of this action is introduced in Ref. [10] and tested numerically in quenched simulations for heavy quarks ($am_h \simeq 0.5$).[2] Very small cut-off effects are observed, and it will be interesting to see unquenched calculations of charm-quark quantities with this new action, which the authors are planning to do.

The bottom quark, however, still has $am_b > 1$ even at the smallest lattice spacings currently available, and therefore cannot be treated with just light-quark methods. Additional input is needed to control effects due to the large b-quark mass. Such input is provided by Effective Field Theory (EFT) in the form of HQET or NRQCD, and all lattice heavy-quark methods (designed to treat b-quarks) involve the use of EFT at some stage in the calculation. The methods differ in how EFT is incorporated, which in turn leads to significant differences in the associated systematic errors between the various methods. In lattice HQET [12] or lattice NRQCD [13] one discretizes the corresponding continuum EFTs. Relativistic heavy-quark actions (Fermilab [14], Tsukuba [15], RHQ [16]) are based on improved versions of the Wilson action supplemented by guidance from HQET for the matching conditions. Finally, two recent methods, which are discussed in more detail in Section 5, generate lattice data with an improved light-quark action and then perform extrapolations to the physical b-quark mass using guidance from HQET. With all methods, a numerical analysis must be based on simulations with multiple lattice spacings to disentangle truncation effects of the heavy-quark expansion from discretization effects and/or to control residual discretization effects.

Given the variety of heavy-quark methods, a new category called "heavy-quark treatment" was developed for FLAG-2. Taking into account the different issues and considerations of each heavy-quark method a set of minimum requirements for the level of improvement (and weak current matching) is specified for each approach. However, no matter what heavy-quark method is used, all calculations must include a careful systematic error study of truncation and/or discretization effects. In addition, FLAG-2 therefore adopts a data-driven quality criterion for "continuum extrapolation" for evaluating lattice calculations of heavy-quark quantities. The purpose of the modified continuum-extrapolation criterion is to measure and evaluate the differences between the data at the finest lattice spacings used in the simulations and the continuum extrapolations. Obviously, large differences would be an indication of large residual cut-off effects. Further, comparing the observed deviations from the continuum extrapolated values with the quoted systematic error estimates provides additional information about the reliability of the systematic error estimates. For a detailed discussion of the FLAG-2 heavy-quark criteria I refer the reader to Section 2.1.2 of Ref. [7].

Finally, I note that all heavy-quark methods that are represented in the phenomenologically relevant results discussed here (and in the FLAG-2 review) have been thoroughly tested, for example via calculations of heavy meson spectra and mass splittings, that can be compared against experimental measurements. A brief description of such tests is given in Section 8 of Ref. [7].

There are now close to a dozen groups who have generated ensembles with $N_f \geq 2$ flavors of dynamical fermions covering a range of lattice spacings, spatial volumes, and light quark masses.

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[2]Here, and throughout this review the subscript h refers to a generic heavy quark, which may be charm or bottom, or a quark with a mass in-between the two.

Most modern simulations use improved actions, and there is a lot of variation especially in the fermion actions used for the quark sea. The first results with physical-mass sea pions were presented by the PACS-CS collaboration [19] with $N_f = 2 + 1$ nonperturbatively improved Wilson flavors on a relatively small ($L \simeq 2$ fm) spatial volume. Since then the BMW [20], MILC [21], and the RBC/UKQCD [22, 104] collaborations have generated ensembles with $N_f \geq 2 + 1$ with physical-mass sea pions on large spatial volumes. Other collaborations also have plans for (or are in the process of) generating ensembles at the physical point. A review of the status of ensemble generation (as of early 2012) is provided in Ref. [18]. Included in this review are the physical-mass PACS-CS and BMW ensembles but the picture is currently changing rather quickly, with two more groups (MILC and RBC/UKQCD) having generated ensembles at the physical point since early 2012, and with more to follow soon. In particular, several results obtained with the new ensembles generated by the MILC and RBC/UKQCD collaborations were presented at (or before) this conference.

The availability of physical-mass ensembles at small lattice spacings and on large spatial volumes represents an important breakthrough for lattice QCD. Chiral extrapolations become *interpolations*, and are used to correct for mis-tunings of the light-quark masses. For that purpose the inclusion of ensembles with larger-than-physical mass pions into an analysis based on physical-mass ensembles is still useful. In summary, where previously the systematic error due to the chiral extrapolation and the truncation of the chiral expansion had a prominent place in the error budget, with physical-mass simulations this error is transformed to a chiral interpolation error and demoted to a much less prominent place among the subleading errors. Not surprisingly, this development has a significant impact on the the overall precision that is now possible, as discussed in the physics sections below.

A final note concerns the fact that almost all lattice groups make their ensembles available to other collaborations. Hence, two different results obtained using different methods may be statistically correlated if they use the same (or overlapping) sets of ensembles. Such correlations are taken into account in the FLAG-2 averages. Indeed, there are a few cases in the heavy-quark sector, where all the results to date have been obtained on overlapping ensembles, and taking the statistical correlations into account is therefore important at present. However, once results for a given quantity are available from many different lattice groups, on different sets of ensembles, such correlations are less significant. This is indeed the case for the kaon decay constant discussed in the next section.

3. Leptonic and semileptonic kaon decays

The focus of this section is on tree-level leptonic and semileptonic decays of kaons into leptonic final states of the form $\ell\nu$, which proceed via the charged-current interaction in the SM. The study of such decays yields information on the CKM element V_{us}. While the semileptonic decay probes the W-boson coupling to up and down quarks via the vector current, the leptonic decay does so via the axial vector current. Hence observing consistency between the two avenues to V_{us} tests the $V - A$ structure of the current. The hadronic matrix elements needed to describe leptonic decays are parameterized in terms of decay constants. Likewise, the hadronic matrix elements relevant to semileptonic decays are parameterized in terms of form factors, see Eq. (4.1) below. Here I discuss
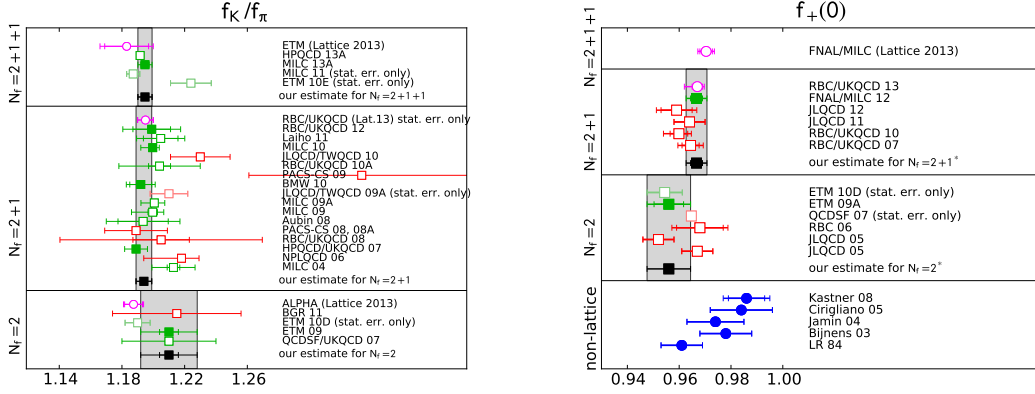
Figure 1: Overview of lattice-QCD results for (the Isospin average) f_K/f_π (left) and for $f_+^{K\to\pi}(0)$ (right) adapted from Ref. [7]. The black data points and grey bands are the FLAG-2 averages. Filled green data points are results included in the respective averages, while unfilled data points are not included. Blue data points (right panel) are non-lattice results. Magenta data points are new results presented at Lattice 2013 not included in Ref. [7] and shown here for comparison purpose only.

the status of lattice-QCD calculations of the ratio of kaon to pion decay constants, f_K/f_π, and of the semileptonic form factor $f_+^{K\to\pi}(0)$.

Typically, when a collaboration generates new ensembles, the ensemble generation often includes "measurements" of some simple quantities, such as pion and kaon masses (and other light hadron masses) and pion and kaon decay constants. The pion and kaon decay constants are therefore among the most studied flavor physics quantities in lattice QCD. This is illustrated in the left panel of Figure 1 for the ratio f_K/f_π which shows a large number of entries.

Naturally, lattice results exist for all values of N_f for which there are ensembles, and there is no discernible difference (within errors) between results obtained with different N_fs. For $N_f = 2+1$ four lattice results [22, 24−26] contribute to the FLAG-2 average, which is obtained with an uncertainty of 0.42%. For $N_f = 2$ and $N_f = 2+1+1$ there are fewer entries, and in each case the FLAG-2 average is obtained from a single lattice result [27, 28]. In the $N_f = 2+1+1$ case, the single lattice result that enters the FLAG-2 average is reported by MILC and is the first to be obtained on physical-mass ensembles [28]. Accordingly, a total uncertainty of 0.4% is quoted, smaller than for any previous result. An independent analysis of essentially the same data set as Ref. [28] is presented by HPQCD in Ref. [29], which however claims a much better precision quoting a total uncertainty of 0.2%. The cause of the surprisingly large error discrepancy is unclear at present. It could possibly be at least in part due to HPQCD using (continuum) NLO chiral perturbation theory (χPT) which may provide better control over systematics than the polynomial interpolations used in Ref. [28]. However, this is likely not the entire story, and further study is certainly needed. In any case, even the larger error quoted by MILC in Ref. [28] is evidence of the positive effect of physical-mass ensembles on the quality of lattice-QCD predictions for this quantity.

For $N_f = 2+1$ first results from physical mass ensembles have also recently appeared. A first preliminary calculation of f_K/f_π on RBC/UKQCD's physical mass ensembles was presented at this conference [23]. The analysis includes a continuum extrapolation and chiral interpolation to the physical light quark masses, but quotes statistical errors only. The first result for f_π on the

$N_f = 2+1$ physical-mass ensembles generated by the BMW collaboration is presented in Ref. [30] with a complete error budget and a total uncertainty of 1%, again smaller than for any previous result. New results for f_K/f_π were also presented by the ETM collaboration [31] for $N_f = 2+1+1$ and by the ALPHA collaboration [32] for $N_f = 2$.

The right panel of Figure 1 summarizes the status of lattice-QCD results for the semileptonic form factor $f_+^{K \to \pi}(0)$. Here too, first results obtained on physical-mass ensembles with both $N_f = 2+1+1$ [33] as well as $N_f = 2+1$ [34] dynamical flavors were presented at the Lattice conference. FNAL/MILC [33] quotes a total uncertainty of 0.33%, again smaller than any previous result. RBC/UKQCD [34] quotes a larger uncertainty (0.5%), because the result doesn't yet include the new physical-mass ensembles, and the uncertainty is therefore still dominated by chiral extrapolation errors. This error will be reduced, once the physical-mass ensembles are included and the analysis is finalized.

As discussed at the beginning of this section, both f_K/f_π and $f_+^{K \to \pi}(0)$ can be used to determine $|V_{us}|$. At present, the lattice-QCD predictions of f_K/f_π and $f_+^{K \to \pi}(0)$, are still the dominant sources of error in the respective $|V_{us}|$ error budgets. But already, Ref. [33] yields a first-row unitarity test where the contribution from the V_{us} term to the total error is slightly smaller than the contribution from the V_{ud} term. Improving the lattice-QCD calculations further is straightforward, a matter of increasing the statistics and/or adding more ensembles. Hence, we can look forward to a time in the near future where the dominant contributions to the V_{us} error budgets will be from experimental uncertainties and from the EM corrections which are currently calculated in χPT.

A final note concerns strong isospin breaking effects. For leptonic kaon decay, the decay constant of the charged kaon, or, in our case f_{K^+}/f_{π^+} is the quantity that is relevant for phenomenology. Some, but not all lattice groups calculate the charged meson decay constants by extrapolating (or interpolating) to the correct valence light-quark mass. This procedure correctly accounts for the leading isospin-breaking effects [35]. Other groups calculate the isospin average f_K/f_π, which then can be converted to f_{K^+}/f_{π^+} either using χPT [36] or with a separate lattice-QCD calculation based on a new method developed in Ref. [37]. Similarly, for semileptonic kaon decay, the form factor of interest concerns $K^0 \to \pi^- \ell^+ \nu_\ell$ decay, which is what essentially all lattice groups calculate. In this case the lattice results include isospin breaking effects at NLO in the chiral expansion. The leading isospin breaking effects due to EM interactions that enter through the light-quark masses can be taken into account by considering EM effects on light meson masses. However, the long-distance radiative corrections to the leptonic or semileptonic decay are estimated phenomenologically [38].

4. Leptonic and semileptonic D-meson decays

The leptonic $D_{(s)}$-meson decay constants and semileptonic D-meson form factors discussed in this section provide determinations of the CKM elements V_{cd} and V_{cs}. As discussed in Section 2, charm-quark quantities can be studied with relativistic heavy-quark actions as well as with improved light-quark actions. Indeed, the $D_{(s)}$-meson decay constants have been calculated with a number of different actions. Within the improved light-quark action category, there are results that use the HISQ action [39 – 41], the twisted-mass Wilson [27, 42, 43, 31] and nonperturbatively improved Wilson fermion actions [44], and the Domain Wall fermion action [45]. Within the rela-
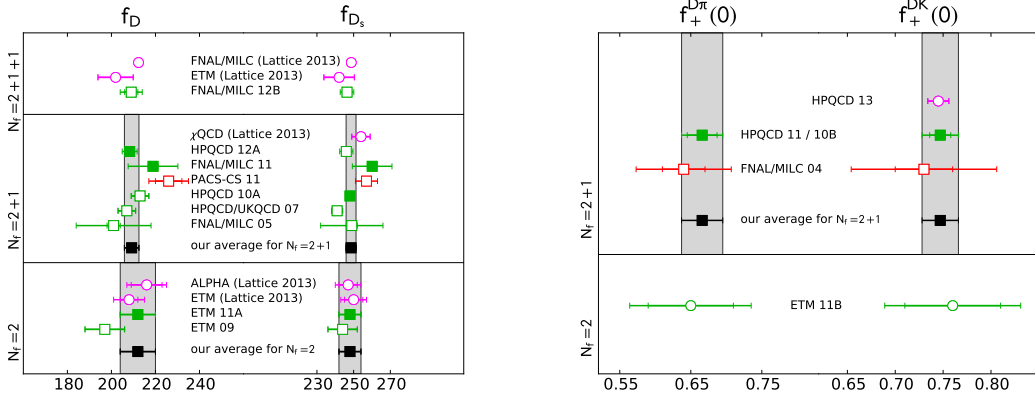
Figure 2: Overview of lattice-QCD results for f_D, f_{D_s} (left) and $f^{D\to\pi}(0)$, $f^{D\to K}(0)$ (right) adapted from Ref. [7]. See the caption of Figure 1 for an explanation of the color coding.

tivistic heavy-quark action category, there are results that use the Fermilab [46] and Tsukuba [47] actions.

The left panel of Figure 2 shows that despite the aforementioned variety, there are fewer independent results that enter the FLAG-2 averages as compared to the kaon decay constant ratio, two for $N_f = 2+1$ [39, 46], one for $N_f = 2$ [42], and none for $N_f = 2+1+1$ (because the result with $N_f = 2+1+1$ [40] is a conference proceedings). For the $N_f = 2+1$ case the FLAG-2 average is quoted with a precision of 1.1% (1.6%) for f_{D_s} (f_D), where the statistical correlations due to the two results using overlapping sets of ensembles are included. However, the left panel of Figure 2 also shows that there is a lot of recent activity, with five new results (shown in magenta) having been presented at the conference. There are two new results with $N_f = 2$ dynamical flavors [43, 44], one with $N_f = 2+1$ [45] and two with $N_f = 2+1+1$ [31, 41]. An update of ongoing work to extend the analysis of Ref. [46] (which uses the Fermilab approach) to the complete set of $N_f = 2+1$ MILC ensembles was also reported at this conference [48]. The analysis, which includes both the D- and B-meson decay constants, is still blind. Therefore no quotable results were presented. The blinding factor will be removed only after the systematic error analysis is final. Finally, a preliminary calculation of $f_{D_{(s)}}$ with $N_f = 2$ Domain Wall fermions was also presented at this conference [49], but again without a quotable result.

The first $D_{(s)}$-meson decay constant results obtained on physical-mass ensembles were presented by the PACS-CS collaboration [47]. However these results do not enter the FLAG-2 averages, since the simulations are performed at one lattice spacing only (and in a small physical volume). Preliminary results on physical-mass ensembles at multiple lattice spacings and on large physical volumes are presented by the FNAL/MILC collaboration [40, 41] using the MILC collaboration's $N_f = 2+1+1$ HISQ ensembles. Ref. [41] is an update of Ref. [40] and includes ensembles at a finer lattice spacing as well as increased statistics on the previously used ensembles. In addition, the new analysis also includes the use of NLO Heavy Meson χPT (HMχPT) for all staggered fermions [50]. With all these improvements, the FNAL/MILC collaboration achieves a very precise result with 0.5% total uncertainty. The analysis is published as a proceedings only, but it does include a complete and extensive error budget. Other factors that contribute to the smallness of the total uncertainty are the use of the highly improved action (HISQ), which yields small discretization errors, and of an absolutely normalized current, which avoids renormalization

errors. Indeed, these factors also contributed to the results obtained by HPQCD in Refs. [39] which previously had the smallest total errors.

A final comment concerns again strong isospin breaking effects, which are also relevant here, given the recent results at sub-percent precision. As for kaon decay, the leading effects can simply be taken into account by extrapolating (or interpolating) to the correct valence light-quark mass. While most lattice groups quote results for the isospin average f_D, the FNAL/MILC collaboration [46, 41, 40] calculates f_{D^+}. The experimental accuracy with which leptonic $D_{(s)}$-meson decay rates are measured suffers from the helicity suppression of the decay. The averages [51] for the experimentally measured branching fractions imply contributions to the respective $|V_{cs}|$ and $|V_{cd}|$ error budgets of 1.8% and 2.4%.

For semileptonic D (and B) meson decays, unlike for semileptonic kaon decay, knowledge of the form factors is desired over the entire kinematically allowed range of momentum transfer. For decays into a pseudoscalar final state, the weak hadronic matrix element can be parameterized in terms of the two form factors f_+ and f_0. Using the process $D \to K\ell\nu$ as an example, we have

$$\langle K|V^\mu|D\rangle = f_+(q^2)\left(p_D^\mu + p_K^\mu - \frac{m_D^2 - m_K^2}{q^2}q^\mu\right) + f_0(q^2)\frac{m_D^2 - m_K^2}{q^2}q^\mu , \qquad (4.1)$$

where V^μ is the appropriate flavor changing vector current and $q^\mu \equiv (p_D - p_K)^\mu$ is the momentum transfer. The SM prediction of the semileptonic decay rate in the rest frame of the parent meson (still using the process $D \to K\ell\nu$ as an example) reads

$$\frac{d\Gamma(D \to K\ell\nu)}{dq^2} = (\text{known}) \, |\mathbf{p}_K|^3 \, |V_{cs}|^2 \, \left|f_+^{D\to K}(q^2)\right|^2 \qquad (4.2)$$

where "(known)" is now just proportional to G_F. In order to determine, say, $|V_{cs}|$, we just need to calculate $f_+^{D\to K}(0)$, since the measured differential decay rate can be extrapolated to $q^2 = 0$. However, lattice calculations of the form factor shape provide a test of the lattice methods via a comparison between theory and experiment. In principle, form factor shapes can then also be used to to obtain better determinations of the corresponding CKM element.

A summary of results for $f_+(0)$ for $D \to K$ and $D \to \pi$ semileptonic decays is provided in the right panel of Figure 2. Currently, only a few independent results exist: three for $N_f = 2+1$ [52–54] and one for $N_f = 2$ [55], where Ref. [53] forms the FLAG-2 average, which is quoted for only $N_f = 2+1$ because the $N_f = 2$ result appears in a conference proceedings. Refs. [52, 54, 55] present results for $f_{+,0}(q^2)$ (normalization and shape). HPQCD calculates $f_0(q^2)$ only in Ref. [53], where the kinematic constraint $f_0(0) = f_+(0)$ is used to obtain the normalization, $f_+^{D\to K}(0)$ ($f_+^{D\to\pi}(0)$), with a total uncertainty of 2.5% (4.4%). A new calculation by HPQCD of the $D \to K$ form factors $f_{+,0}(q^2)$ is presented in Ref. [54] for $N_f = 2+1$, where a first attempt is made to determine $|V_{cs}|$ via a shape analysis that combines the lattice results with experimental measurements over the entire kinematic range. For $N_f = 2$ a new calculation of the $D \to K$ and $D \to \pi$ form factors covering the entire kinematic range was presented by the ETM collaboration [56] at this conference, without, however, quotable results. Before discussing the methods used in these two recent works in more detail, a few general comments are needed.

As in any lattice-QCD calculation, the lattice form factors are calculated over a range of lattice spacings and light-quark masses, and must be extrapolated to the continuum and extrapolated (or

interpolated) to the physical light-quark masses. But in this case, one generates data for a set of recoil momenta in order to obtain the form factors over a certain kinematic range of recoil energies and there is more than one way for performing the chiral-continuum extrapolation. A theoretically well motivated analysis strategy using a two-step procedure is outlined first. It starts with HMχPT for semileptonic decays [58 – 60], which, if supplemented with appropriate lattice-spacing dependent discretization terms, provides a convenient framework for a combined chiral and continuum extrapolation/interpolation. With HMχPT one parameterizes the matrix elements in terms of the form factors ($f_{v,p}(E)$) [58] that are functions of the recoil energy and from which one can construct (the chiral-continuum extrapolated) $f_{+,0}(q^2)$.

In the second step, a model-independent parameterization of the form factor shape can be obtained with so-called z-expansions, which are based on analyticity and unitarity. First the momentum transfer q^2 is mapped to a new variable z, via

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}} \; , \qquad (4.3)$$

where $t_+ = (m_D + m_K)^2$ and $t_0 < t_+$ is chosen for convenience. The form factors can then written as a series expansion in z,

$$f(q^2) = \frac{1}{B(q^2)\Phi(q^2, t_0)} \sum_n a_n z^n \; , \qquad (4.4)$$

where $B(q^2)\Phi(q^2, t_0)$ contains poles and cuts, and where one can derive bounds on the sum of the squares of the coefficients a_n. Different versions of z-expansions differ in the choice of prefactors $B(q^2)\Phi(q^2, t_0)$ and in how constraints are included in the coefficients. The two most common choices are referred to as BGL [61] and BCL [62]. In summary, in the two-step procedure one obtains a model-independent parameterization of the form factors over the desired kinematic range, where one first uses HMχPT to perform the chiral-continuum extrapolation, and then feeds the results into z-expansion fits.

In Ref. [53] the HPQCD collaboration introduced a modified z-expansion, replacing the two-step analysis strategy discussed above with a simpler one-step method, however, at the cost of adding ad-hoc assumptions about the lattice-spacing, light-quark mass, and recoil-energy dependence of the z-expansion coefficients. Instead of first performing a chiral-continuum extrapolation, they fit their lattice form-factor data to a BCL-like function, where the z-expansion coefficients, a_n are allowed to depend on the light-quark mass and on the lattice spacing using terms that are inspired by, but not derived from χPT and Symanzik EFT. I think that further tests are needed to understand whether or not the various modifications that are currently used properly account for the systematic effects. However, a derivation of the modified z-expansion from the EFTs would be even better.

Ref. [53] uses the modified z-expansion to perform a simultaneous chiral-continuum extrapolation of f_0 coupled with an interpolation to $q^2 = 0$. The modifications to the z-expansion coefficients include light-quark mass dependent terms, generic a^2 terms, heavy-quark discretization terms, and recoil energy dependent terms. In Ref. [54] HPQCD extends the analysis of Ref. [53] to $f_{+,0}(q^2)$ over the full kinematic range. They use a subset of the ensembles included in Ref. [53] (three instead of five), but increase the statistics on each ensemble by a factor of three or more among other

improvements. The modified *z*-expansion used in Ref. [54] is more restricted with only light-quark mass and generic lattice-spacing terms added to the coefficients. The quoted result for $f_+^{D \to K}(0)$, with a total uncertainty of 1.5% is a little less than a factor of two more accurate than the previous result, but it is unclear what role the more restricted *z*-fit function has in this reduction of the error. The HPQCD collaboration also presented results for the $D_s \to \phi \ell \nu$ form factors [57]. With a vector-meson final state four form factors are needed to parameterize the matrix elements, and one can measure and predict three angular distributions in addition to the q^2 distribution. Ref. [57] uses the same set of ensembles and the modified *z*-expansion with the same lattice-spacing and light-quark mass dependence as Ref. [54].

The ETM collaboration [56] presents results for the $D \to K(\pi)$ form factors from two different analysis strategies. In the first, they perform a chiral-continuum extrapolation of the form factors at fixed q^2, where the lattice form factor data are first interpolated to a fixed set of q^2 values using a parabolic spline interpolation and then extrapolated to the physical light-quark masses and to the continuum using terms linear in m_π^2 and in a^2. In the second, they use the modified *z*-expansion where terms constant and linear in *z* are kept, and where the coefficients are extrapolated using the same terms linear in a^2 and m_π^2 as in the first method. The results they obtain from the two methods are consistent within the quoted errors.

An almost final comment concerns strong isospin breaking and EM effects. For semileptonic *D*- (and *B*-) meson decays, the experimental results are obtained from an average of charged and neutral meson decays, so lattice calculations of the isospin averages are appropriate here. However, the current HFAG averages do not take the EM correction (of order 2%) due to the Coulomb attraction between the charged final states in neutral meson semileptonic decays into account.

Finally, I comment on how form factor results are presented, and what is needed when a calculation quotes more than just one number (and error budget). If a work presents results for both normalization and shape, it also should provide the corresponding covariance matrix, in addition to a complete error budget, so that the results can be properly combined with other lattice calculations and with experimental results.

5. Leptonic and semileptonic $B_{(s)}$-meson decays

The leptonic *B*-meson decay constants and semileptonic *B*-meson form factors discussed in this section provide determinations of the CKM elements V_{ub} and V_{cb}. However, the quantities discussed in this section can also be used to obtain SM predictions of rare processes, such as the $B_s \to \mu^+ \mu^-$ decay already discussed in the introduction, that are sensitive to contributions from new physics and can therefore be used to constrain BSM theories.

Before presenting the lattice results, I would like to discuss two relatively new heavy-quark methods that use light-quark actions (with $O(am_h)^2$ errors) in calculations of *b*-quark quantities. The ETM collaboration has developed an interesting strategy, aptly named the "ratio method" [63], in which they form ratios of physical quantities such that the static limit of the ratios is equal to unity and where discretization errors are suppressed by the step-scaling factor used in constructing the ratios. Using decay constants as an example, and denoting the decay constant for a meson with heavy quark flavor *h* and light valence quark flavor ℓ as $f_{h\ell}$, and the corresponding heavy-meson mass as M_H, it is well known that the combination $\phi(m_h) \equiv f_{h\ell}\sqrt{M_H}$ has a well-defined static

limit ($m_h \to \infty$).[3] Then the ratio $z_{h\ell}(m_h) \equiv \frac{\phi(m_h)}{\phi(m_h/\lambda)}$ (with λ a fixed step-scaling parameter) has the property that $z(m_h \to \infty) \to 1$. The z-ratios have the further property that heavy-quark discretization errors are suppressed with a suitable choice of step-scaling factor λ. ETM then uses the twisted-mass Wilson action to calculate the z-ratios for a series of fixed heavy-quark masses (separated by the step scaling factor $1 < \lambda \lesssim 1.3$) starting from the charm quark mass up to some heavy-quark mass while keeping $am_h < 1$. In addition they calculate the physical quantity of interest (ϕ in this example) at the charm-quark mass. After chiral-continuum extrapolation of the z-ratios (and $\phi(m_c)$) they then interpolate the z-ratios to the physical b-quark mass using guidance from HQET, and finally construct the quantity of interest (f_B in this case) from the step-scaled product of z-ratios and $\phi(m_c)$. This procedure, outlined here for the case of heavy-meson decay constants can be applied to any heavy-quark quantity that has a well-defined static limit. Indeed, ETM has used this method in determinations of the b-quark mass, as well as in calculations of $B_{(s)}$-meson decay constants and of $B_{(s)}$-meson mixing parameters [43, 64] (see Section 6). In addition the Orsay group has used this method for calculations of semileptonic form factors for $B_{(s)} \to D_{(s)}^{(*)}$ decays [65 – 67] . Finally, I note that ETM's ratio method can be used with any light-quark action that has $O(a^2)$ discretization errors. The cancellation of heavy-quark discretization errors observed in the z-ratios is a generic feature of the ratios themselves, which I would expect to be largely independent of the underlying light-quark action.

Another interesting method, presented by the HPQCD collaboration in Ref. [68], is dubbed the "heavy HISQ" method. This method takes advantage of the fact that the leading heavy-quark discretization errors are suppressed in the HISQ action [69] compared to other light-quark actions, a fact which has been tested extensively in numerical simulations [70]. The HISQ action, like the Asqtad action, is essentially a tadpole-improved staggered action (albeit with much reduced taste-violating effects), which means that the leading heavy-quark discretization errors are $O(\alpha_s(am_h)^2, (am_h)^4)$. However, in the construction of the HISQ action, the leading heavy-quark discretization errors are further reduced by factors of v/c via an adjustment of the Naik term, yielding errors of the form $\alpha_s(v/c)(am_h)^2, (v/c)^2(am_h)^4$. Another important ingredient in this approach, both for studying and controlling discretization effects and for reaching large enough heavy-quark masses (in physical units), is the availability of ensembles that cover a large range of lattice spacings and reach down to very small lattice spacings. The HPQCD collaboration used the MILC collaboration's Asqtad ensembles, which include lattice spacings in the range $a \approx 0.045 - 0.15$ fm. In Ref. [68] the f_{B_s} decay constant is calculated from relativistic data that include heavy-quark masses as large as $am_h \lesssim 0.85$. Surprisingly they find that discretization errors (as measured by the observed deviations from the continuum fit line) decrease with increasing heavy-meson mass M_H (in physical units). They perform a combined continuum extrapolation and HQET-inspired heavy-quark extrapolation to the physical b-quark mass. The heavy HISQ method has also been used in studies of heavy-light and heavy-heavy meson spectra and decay constants [70] as well as current-current correlators [71]. Chiral extrapolations are avoided in these calculations by focusing on heavy-quark quantities with strange (or heavier) valence quarks where sea-quark mass dependence is a subdominant source of error (at the current level of precision).

The left panel of Figure 3 shows that the status of B-meson decay constant calculations is

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[3]In their main analysis, ETM actually uses the heavy-quark pole mass instead of the heavy-meson mass for ϕ.
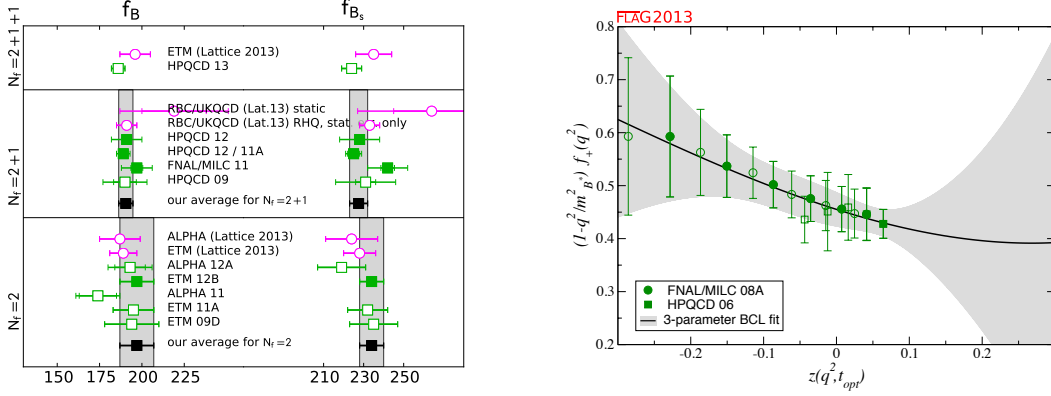
Figure 3: Overview of lattice-QCD results for f_B, f_{B_s} (left) and $(1 - q^2/m_{B^*}^2)f_+^{B \to pi}(q^2)$ (right). The plot in the left panel is adapted from Ref. [7] while the right panel is taken directly from Figure 20 of Ref. [7]. See the caption of Figure 1 for an explanation of the color coding in the left panel. For the right panel, filled symbols are lattice data included in the 3-parameter BCL fit, open symbols are not. The FLAG-2 fit curve is shown as a black line with grey error band.

similar to the case of D mesons, both for considering the number of lattice results that enter the FLAG-2 averages for $f_{B_{(s)}}$, and for considering the number of new results presented at this conference. Another similarity is that the B-meson decay constant results are obtained with a variety of different heavy-quark methods, including the ratio method (with twisted-mass Wilson fermions) [73, 43, 64], the heavy-HISQ method [68], lattice HQET [77], lattice NRQCD [72, 74], Fermilab heavy quarks [46], the RHQ action [75], and the static limit [76]. However, the uncertainties on both the individual lattice results and on the FLAG-2 averages are larger in the case of B-meson quantities, reflecting the larger discretization/truncation errors and also the generally larger statistical errors. Three results enter the FLAG-2 averages for $N_f = 2 + 1$ [46, 68, 72] , one for $N_f = 2$ [73], and none for $N_f = 2 + 1 + 1$. For the $N_f = 2 + 1$ case the FLAG-2 average is quoted with a precision of 2.0% (2.2%) for f_{B_s} (f_B), where the statistical correlations due to the three results using overlapping sets of ensembles are included. Five new results (shown in magenta in the left panel of Figure 3) were presented at this conference, two with $N_f = 2$ [43, 77], two with $N_f = 2 + 1$ [75, 76] and one with $N_f = 2 + 1 + 1$ [64].

The first $B_{(s)}$-meson decay constant results obtained on physical-mass ensembles are presented by the HPQCD collaboration in Ref. [74] on a subset of the MILC collaboration's $N_f = 2 + 1 + 1$ HISQ ensembles. However, here, the gain from the use of physical-mass ensembles is less apparent, since there typically are other sources of uncertainty that dominate (or significantly contribute to) the error budget. In this particular case, the dominant error is due to the renormalization of the NRQCD-HISQ current, calculated at one-loop in mean-field improved perturbation theory. A comparison of the error budgets between this result and HPQCD's previous calculation with NRQCD-HISQ valence quarks [72] performed on a set of Asqtad ensembles over roughly the same range of lattice spacings, shows a reduction in the total uncertainty by roughly a factor of two. However, a closer look at the error budgets reveals that the biggest change is in the perturbative error estimates. Since the valence-quark actions and the lattice spacings are the same in both cases, so too are the perturbative renormalization factors the same (up to small differences due to the masses being slightly different in the two cases). However, Ref. [74] quotes a perturbative uncertainty

that is smaller by almost a factor of three. They justify this error reduction by reorganizing the perturbative expansion in a manner consistent with the mostly NPR method [78].

Because of the helicity suppression the only experimentally measured leptonic B-meson decay rate is for the $B \to \tau \nu_\tau$ channel. The branching ratio measurements still have rather large errors and the experimental average of Ref. [51] implies a contribution to the $|V_{ub}|$ error budget of around 10%.

For semileptonic decays of $B_{(s)}$ mesons, we distinguish between light final states (heavy-to-light case) and charm final states (heavy-to-heavy case), which we will discuss in turn. For the heavy-to-light case, we further distinguish between tree-level weak decays (such as $B \to \pi \ell \nu$ or $B_s \to K \ell \nu$) that are relevant to $|V_{ub}|$ determinations and rare (or FCNC) decays (such as $B \to K^{(*)} \ell^+ \ell^-$) that are used to constrain BSM theories. The SM decay rates for semileptonic B-meson decays to τ-leptons receive a contribution from the scalar form factor $f_0(q^2)$ as well as the usual $f_+(q^2)$, while rare semileptonic B-meson decay rates also receive contributions from the tensor form factor $f_T(q^2)$. For semileptonic B-meson decays to light hadrons the recoil momentum range accessible to lattice-QCD calculations is significantly smaller than the full kinematic range of the decay. Furthermore, experimental measurements of differential decay rates are most accurate in the high recoil momentum region not directly accessible to lattice QCD. Model-independent parameterizations of the form factor shape are essential for obtaining information about the form factors outside the lattice-data region and all modern lattice-QCD calculations use z-parameterizations for just that purpose.

The FLAG-2 review includes averages only for $N_f = 2 + 1$ and only for the $B \to \pi$ channel at present, since this was the only channel with published lattice results at the time the review was written. The right panel of Figure 3 illustrates the status of $B \to \pi$ form factor calculations prior to this conference. The two calculations [79, 80] that are included in the FLAG-2 form factor average are both over five years old. The first [79] uses lattice NRQCD to treat the b quark while the second [80] is based on the Fermilab approach. Only Ref. [80] quotes the covariance matrix for the form factor results, and the FLAG-2 average is therefore obtained from combining the data from Ref. [80] with one data point from Ref. [79]. Since the two results use overlapping sets of ensembles statistical correlations are included in the average. FLAG-2 performs a (constrained) BCL fit to the combined lattice data, which is shown as a grey band in the right panel of Figure 3 and then further combines the lattice data with experimental data from BaBar [81] and Belle [82], without, however, combining both experiments. This procedure yields an uncertainty on $|V_{ub}|$ in either case of about 6%, where the longstanding tension between the $|V_{ub}|$ determinations from exclusive and inclusive semileptonic B decays still stands at about 3σ. The good news is that a number of new calculations for this channel are underway both for the $N_f = 2 + 1$ [83–85] and $N_f = 2$ [86] cases. Further good news is that each of these projects is based on a different heavy-quark action, including Fermilab, NRQCD, RHQ, and HQET. Some groups [84, 87] are also now calculating the form factors for $B_s \to K \ell \nu$ for $N_f = 2 + 1$. This process has not yet been measured experimentally. Once it is, it will provide an alternate determination of $|V_{ub}|$. All the new results (or calculations in progress) described here use the theoretically well-motivated two-step analysis strategy outlined in the previous section, which allows for complete control over the dominant systematic errors over the full kinematic range, provided, of course, that the underlying numerical simulations cover a large enough range of lattice spacings and light-quark masses.
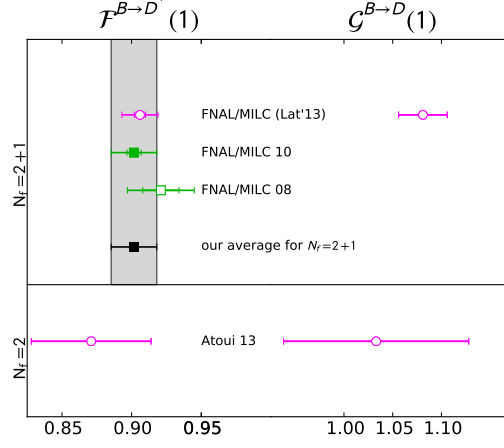
Figure 4: Overview of lattice-QCD results for $\mathscr{F}^{B \to D^*}(1)$, $\mathscr{G}^{B \to D}(1)$ adapted from Ref. [7]. See the caption of Figure 1 for an explanation of the color coding.

Rare semileptonic decays have also received attention recently. The first published $N_f = 2 + 1$ results for the rare decay $B \to K \ell^+ \ell^-$ form factors appeared by the HPQCD collaboration this summer [88], where the form factors are calculated with $\approx 5 - 9\%$ uncertainty in the low-recoil region. Another calculation of this process by the FNAL/MILC collaboration is also close to completion [89, 87]. Ref. [88] compares the z-fit results for the form factors from the two-step analysis to results obtained from a modified z-expansion fit (see Section 4). Overall consistent behavior between the two fit bands is seen, even in the extrapolated region, but the coefficients of the two z-fits are not compared in detail. Ref. [88] (second paper) also provides the SM prediction for the differential decay rate implied by HPQCD's form factor results which is in turn compared to experimental measurements showing overall good agreement.

Rare semileptonic B-meson form factors for decays to vector final states, i.e. $B \to K^* \ell^+ \ell^-$ and $B_s \to \phi \ell^+ \ell^-$, are calculated by the Cambridge group in Ref. [90]. With vector final states there are a number of interesting observables that can be studied in detailed comparisons between theory and experiment. The second paper in Ref. [90] provides SM predictions as well as constraints on the BSM contributions to the Wilson coefficients C_9 and C_9'. As is well known, the lattice-QCD methods currently used for studying weak decays do not properly treat final states that are resonances, such as the K^* and ϕ mesons. An exception is the D^* meson (see below), where χPT guides the chiral extrapolation to the physical light-quark masses. The calculation of Ref. [90], while providing a study of other systematic errors, doesn't include an estimate of this source of error. One can reasonably argue that the small width of the ϕ meson may imply a smaller systematic effect in the $B_s \to \phi$ channel, but without further study to quantify the effect one has to keep in mind that the SM predictions and BSM constraints derived from such lattice form factor calculations contain an unknown systematic error. This situation is, however, no worse than for phenomenological approaches to form factor calculations based, for example, on light-cone sum rules [91].

The current FLAG-2 average for the $B \to D^*$ form factor at zero recoil (available only for $N_f = 2 + 1$ at present) is based on a series of calculations by the FNAL/MILC collaboration [92] with Fermilab bottom and charm valence quarks. The results are summarized in Figure 4. The observed reduction in error from the initial result in Ref. [92] quoted with a 2.6% uncertainty to the most

recent result presented in Ref. [93] with a quoted uncertainty of 1.4% reflects improvements due to adding ensembles at smaller lattice spacings, with smaller light-quark masses, and with increased statistics. The final result of this series of calculations presented in Ref. [93] is based on essentially the full set of Asqtad ensembles with lattice spacings in the range of $a \approx 0.045 - 0.15$ fm. The finite (but small width) of the D^*-meson can be treated within the framework of HMχPT, and yields a cusp in the continuum-extrapolated chiral-fit curve.

The lattice-QCD result for the $B \to D^*$ form factor can be used to determine $|V_{cb}|$ from the experimental average [9] with a total uncertainty of 1.9%. The experimental average in Ref. [9] does not correct for the Coulomb attraction between the charged final state particles in the neutral B-meson decays. At this level of precision such corrections need to be included, and this is done in the V_{cb} determination reported in Ref. [93]. The longstanding tension between the determination of V_{cb} from exclusive and inclusive decays currently stands at about 3σ. Given this tension, and given the overall importance of V_{cb} in normalizing the unitarity triangle and as a parametric source of uncertainty in the SM model predictions of processes such as $B_s \to \mu^+ \mu^-$, $K \to \pi \nu \bar{\nu}$, and neutral kaon mixing (ε_K), it is important to have additional lattice-QCD calculations, both of the same form factor and of the related $B \to D$ form factor, preferably using different heavy-quark methods. Fortunately there are three new efforts at various stages of completion, two of them independent of Ref. [92]. However, more would be welcome.

The Orsay [65] reports a very preliminary result for the $B \to D^*$ form factor shown in Figure 4. They use the ratio method and the twisted-mass Wilson action to calculate the form factor at two lattice spacings on $N_f = 2$ ensembles generated by the ETM collaboration. The main focus of this study is on the form factors for $B \to D^{**}$ decays (i.e. to orbitally excited D-mesons) [66], which may play a role in resolving the tension between exclusive and inclusive V_{cb} determinations. The reported $B \to D^*$ form factor result provides a consistency check of the method used to determine the $B \to D^{**}$ form factors. The Orsay group [67] also presents a calculation of $B_{(s)} \to D_{(s)}$ form factors using again the ratio method with twisted-mass Wilson fermions and using $N_f = 2$ ensembles generated by the ETM collaboration at four different lattice spacings in the range $a \approx 0.054 - 0.098$ fm. They calculate $\mathcal{G}^{B_s \to D_s}(1)$ ($\mathcal{G}^{B \to D}(1)$) with an accuracy of about 4% (9.2%) and also determine the scalar and tensor form factors near zero recoil. The scalar form factor contributes to semileptonic decays to τ-lepton final states and both form factors may provide constraints on BSM models. The Orsay group finds results that are consisted with those reported in Ref. [95].

Ref. [93] presents a $N_f = 2 + 1$ calculation of the $B \to D$ form factors at non-zero recoil. It uses the same valence-quark actions and the same ensembles as the $B \to D^*$ form factor analysis. The $B \to D$ form factors are calculated for the full kinematic range using a two-step analysis (as described in Section 4) with a BGL z-expansion fit. A complete error budget is provided, and V_{cb} is determined from a combined z-fit of the lattice form factors together with experimental data by BaBar [94] with a total uncertainty of about 5%. The analysis was kept blind until the systematic error budget was final, and the result for V_{cb} is nicely consistent with the one from the $B \to D^*$ analysis, albeit less precise. The HPQCD collaboration is in the process of calculating the $B_{(s)} \to D_{(s)}$ form factors using NRQCD b quarks and HISQ charm quarks [84]. In another new effort [96], it is planned to use the highly improved OK action [97] for a calculation of the $B \to D^{(*)}$ form factors since heavy-quark discretization errors are a dominant source of uncertainty with the

Fermilab method. This project has just been started.

6. Neutral (K, D, and $B_{(s)}$) meson mixing and nonleptonic kaon decay

Neutral meson mixing, being loop-induced in the SM, plays an important role in determining the CP violating parameters of the SM as well as in providing constraints on BSM theories. In the SM, neutral meson mixing receives contributions from hadronic matrix elements of $\Delta F = 2$ local operators, (where $F = B, C, S$). For neutral kaons and $B_{(s)}$-mesons these are the dominant contributions to the SM predictions of the corresponding mixing observables (ε_K, $\Delta m_{d(s)}$). In BSM theories, additional $\Delta F = 2$ local operators can contribute, and the most general $\Delta F = 2$ effective hamiltonian can be written in terms of five operators,

$$\mathcal{H}_{\text{eff}} = \sum_{i=1}^{5} C_i Q_i, \tag{6.1}$$

where the integrated out high-momentum physics is collected into the Wilson coefficients, C_i, which therefore depend on the underlying theory (SM or BSM). In a commonly used basis the local $\Delta F = 2$ operators Q_i take the form

$$Q_1 = \left(\bar{h}\gamma_\mu L q\right)\left(\bar{h}\gamma_\mu L q\right), \qquad Q_2 = \left(\bar{h}L q\right)\left(\bar{h}L q\right), \tag{6.2}$$

$$Q_3 = \left(\bar{h}^\alpha L q^\beta\right)\left(\bar{h}^\beta L q^\alpha\right), \quad Q_4 = \left(\bar{h}L q\right)\left(\bar{h}R q\right), \quad Q_5 = \left(\bar{h}^\alpha L q^\beta\right)\left(\bar{h}^\beta R q^\alpha\right),$$

where $h = b, c, s$ denotes a bottom, charm, or strange quark, $q = d, s, u$ denotes a light (down, strange, or up) quark, and $R, L = \frac{1}{2}(1 \pm \gamma_5)$. The superscripts α, β are color indices, which are shown only when they are contracted across the two bilinears. Hadronic matrix elements of these local operators are calculable with current lattice-QCD methods with high precision.

The SM prediction for the neutral B_q-meson ($q = d, s$) mass difference is given by

$$\Delta m_q = (\text{known}) M_{B_q} |V_{tq}^* V_{tb}|^2 f_{B_q}^2 \hat{B}_{B_q} \tag{6.3}$$

where "(known)" includes short-distance QCD and EW corrections, M_{B_q} is the mass of the B_q-meson, and where $f_{B_q}^2 \hat{B}_{B_q}$ parameterizes the hadronic matrix element of Q_1 in terms of the decay constant f_{B_q} and bag parameter $\hat{B}_{B_q}$ (see, for example, Section 8 of Ref. [7] for further details). In summary, for neutral B_q-meson mixing lattice QCD needs to provide both the decay constants and the bag parameters, or their combination. The ratio of the B_s and B_d mass differences is of particular interest due to the cancellation of statistical and systematic errors in the corresponding ratio of matrix elements:

$$\frac{\Delta m_s}{\Delta m_d} \frac{M_{B_d}}{M_{B_s}} = \left|\frac{V_{ts}}{V_{td}}\right|^2 \xi^2 \qquad \text{with} \quad \xi^2 \equiv \frac{f_{B_s}^2 \hat{B}_{B_s}}{f_{B_d}^2 \hat{B}_{B_d}}. \tag{6.4}$$

A similar, but slightly more complicated expression in comparison to Eq. (6.3) holds for the SM prediction of ε_K (see, for example, Section 6 of Ref. [7]), but only the bag parameter $\hat{B}_K$ is needed in this case. For neutral D-meson mixing long-distance effects in the form of matrix elements of insertions of two effective weak $\Delta C = 1$ operators are an important, if not dominant, contribution to
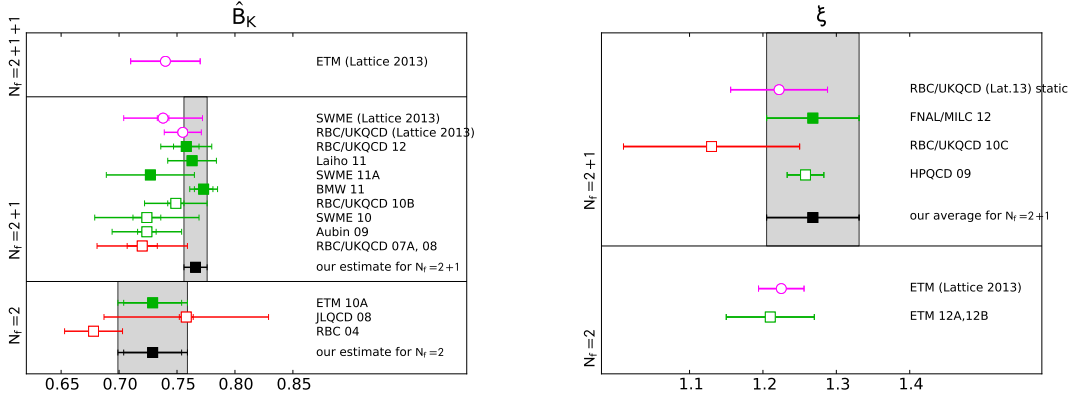
Figure 5: Overview of lattice-QCD results for B_K (left) and ξ (right) adapted from Ref. [7]. See the caption of Figure 1 for an explanation of the color coding.

the SM prediction. Similar long-distance effects (albeit with two $\Delta S = 1$ insertions) also affect the kaon observables Δm_K and ε_K, where, however, they contribute only a small correction to the latter observable. In the kaon case, first results implementing a strategy for the calculation of such effects have already been reported, but for D mesons the matrix elements of such long-distance operators cannot be calculated with current lattice-QCD methods. Hence the current goal of lattice-QCD calculations of D-meson mixing parameters is not to obtain information on the CKM parameters, but to help constrain and discriminate between BSM theories [98].

Most of the lattice calculations from previous years discussed in the FLAG-2 review have focused on hadronic matrix elements of Q_1, needed for SM predictions. While it is now common practice to calculate the matrix elements of all five operators, recent results for the matrix elements of Q_{2-5} are not yet included in the FLAG-2 review. The two panels of Figure 5 provide an overview of the status of lattice-QCD calculations for $\hat{B}_K$ and for ξ.

The left panel of Figure 5 shows that a large effort has gone into lattice-QCD calculations of $\hat{B}_K$. Four results [22, 99 – 101] enter the FLAG-2 average for $N_f = 2 + 1$, which is quoted with an uncertainty of 1.3%, while one result [102] enters the average for $N_f = 2$. Three new results were reported at this conference, two for $N_f = 2 + 1$ [103, 104] and one for $N_f = 2 + 1 + 1$ [105]. The first result from physical-mass ensembles was obtained by the BMW collaboration [101]. It is the most precise result for $\hat{B}_K$ to date with a quoted uncertainty of 1.5% and therefore dominates the $N_f = 2 + 1$ FLAG-2 average. A calculation of $\hat{B}_K$ on RBC/UKQCD's new physical mass ensembles was presented at this conference [104] with a 2.1% uncertainty. The effort of the lattice community to calculate $\hat{B}_K$ has already paid off, since its contribution to the ε_K error budget ranks third, behind the parametric V_{cb} uncertainty and also behind the uncertainty on the short-distance Wilson coefficient (η_{cc}) calculated at NNLO in perturbative QCD. In time, we can expect even smaller errors on $\hat{B}_K$. As already mentioned above, there is also considerable recent activity for calculations of matrix elements of the other four operators, with three groups reporting new results for the additional bag parameters [105 – 107].

A final comment concerns the relevance of the calculation in Ref. [105] of $\hat{B}_K$ using ETM's $N_f = 2 + 1 + 1$ ensembles for the SM prediction of ε_K. The SM prediction of ε_K in terms of a hadronic matrix element of the local $\Delta S = 2$ operator Q_1 is obtained after integrating out the high-momentum degrees-of-freedom, including the charm quarks. Indeed, the relatively low matching

scale produced by this procedure plays an important role in the relatively large uncertainty for the corresponding Wilson coefficient (η_{cc}). With charm quarks present in the $N_f = 2 + 1 + 1$ sea a complete calculation requires the inclusion of the matrix element of the corresponding long-distance operator with two insertions of $\Delta S = 1$ operators into the calculation. However, as I already mentioned, no significant differences between lattice results obtained with different values of N_f have been observed at the current level of precision, and $\hat{B}_K$ is no exception. While the inclusion of charm quarks in the sea is unlikely to have a noticeable effect on $N_f = 2 + 1 + 1$ results for $\hat{B}_K$ in practice, this issue is the reason why these results are not included in the FLAG-2 review at present. Finally, I note that for pretty much every other quantity, calculations that include dynamical charm quarks are "better" in the sense that such calculations include an effect that is neglected (or included at most perturbatively) in calculations based on $N_f = 2 + 1$ simulations, however small this effect actually is.

Since $\hat{B}_K$ is now known with near 1% precision, some lattice groups have turned their attention to the more difficult problems of calculating nonleptonic decay amplitudes as well as long-distance corrections to kaon mixing quantities. Remarkable progress towards understanding the famous $\Delta I = 1/2$ rule for nonleptonic kaon decay is evident from the recent results reported at this conference. The calculations are based on the Lellouch-Lüscher method [108], which requires large physical volumes and physical pion masses. The RBC/UKQCD collaboration [109] presented a result for the $\Delta I = 3/2$ channel on physical-mass ensembles with Domain-Wall fermions at three different lattice spacings. They provide a complete error budget and quote a preliminary total error of an impressive 11%. Their calculation of the $\Delta I = 1/2$ channel is still ongoing. In addition, Ref. [110] presents a first attempt of a calculation of the $\Delta I = 3/2$ and $\Delta I = 1/2$ amplitudes using nonperturbatively improved Wilson fermions. Another focus of "beyond-easy" calculations in the kaon system are the long-distance corrections to Δm_K and ε_K, for which first results were reported by the RBC/UKQCD collaboration at this conference [111].

Turning now to the *B*-meson system, the right panel of Figure 5 shows that for neutral *B*-meson mixing much fewer calculations exist. Prior to this conference three results for $N_f = 2 + 1$ had been reported, one using lattice NRQCD [112], another based on the static limit [113], and the third obtained with Fermilab *b* quarks [114, 115], with only Ref. [114] entering the FLAG-2 average. For $N_f = 2$ the ETM collaboration [73] reported at Lattice 2012 on the results of their first calculation of *B*-meson mixing parameters using the ratio method. Concerning the new results reported at this conference, RBC/UKQCD [76] presents an update of Ref. [113] with the inclusion of ensembles at an additional lattice spacing and also at lighter sea-quark masses. Similarly, Ref. [43] presents the ETM collaboration's final results for *B*-meson mixing quantities on their $N_f = 2$ ensembles. As for kaon and *D*-meson mixing, two recent *B*-meson mixing analyses include calculations of hadronic matrix elements of Q_{2-5} [43, 115]. In summary, while the current status of $B_{(s)}$-mixing calculations is not as good as for decay constants, the level of recent activity suggests that we can expect significant improvements in the near future. In particular, given how much room there is for improvement, it may be that in the future the best SM prediction for the $B_s \to \mu^+ \mu^-$ decay rate comes from the indirect method proposed in Ref. [3], where the decay rate is normalized by Δm_s, and where the lattice input is the B_s bag parameter.

There are several recent efforts for lattice-QCD calculations of the *D*-meson mixing matrix elements for all five local operators. The ETM collaboration has reported on results obtained with

twisted-mass Wilson fermions from both their $N_f = 2$ [43, 73] and their $N_f = 2 + 1 + 1$ [105] ensembles. An ongoing effort to calculate the complete set of *D*- and *B*-meson mixing matrix elements on the full Asqtad set of ensembles with Fermilab bottom and charm quarks is reported by FNAL/MILC in Ref. [116]. Studies of the important long-distance correction to SM *D*-meson mixing or of nonleptonic *D*-meson decay require new lattice methods. The first steps towards developing such methods have recently been taken [117].

7. Conclusions and Outlook

The investment made by a growing number of lattice collaborations to produce ensembles with light quarks simulated at their physical masses and with large physical volumes is in part responsible for the remarkable progress that has been made in quark flavor physics calculations in recent years. The impact of the availability of physical-mass ensembles is most apparent in the kaon system, where results for $\hat{B}_K$, f_K, and f_π are reported with uncertainties close to or below 1% and where the $SU(3)$ breaking quantities f_K/f_π and $f_+^{K\to\pi}(0)$ are now known to better than 0.5%. Remarkably, the contribution of $\hat{B}_K$ to the ε_K error budget is in third place. Quite appropriately then, a significant effort is now dedicated towards calculations of nonleptonic kaon decay amplitudes as well as long distance corrections to ε_K and Δm_K, where physical-mass pions and large physical volumes are an essential ingredient. First results with complete error budgets at the 10% level have now appeared for the easier case, the $\Delta I = 3/2$ amplitude. More results will no doubt follow soon. The contributions from f_K/f_π and $f_+^{K\to\pi}(0)$ to the respective $|V_{us}|$ error budgets will very soon be smaller than those from other sources with the simulations and computations that are under way. The leading strong (and EM induced) isospin breaking effects are already included in these calculations. With further improvements the inclusion of subleading corrections via simulations with non-degenerate light sea quarks (using, for example, reweighting techniques) and also via adding QED to the simulations are all part of the overall program. However, the calculation of the long-distance radiative corrections, which are currently estimated using χPT, requires new methods.

In the *D*- and *B*-meson systems, results from physical-mass ensembles have so far been reported only for decay constants. In the next few years physical mass ensembles will certainly find their way into calculations of *D*- and *B*-meson semileptonic decay form factors and *B*-meson mixing quantities. The recently reached sub-percent accuracy in *D*-meson decay constant results demonstrates the potential effect that physical-mass ensembles can have in reducing the errors in the charm sector. But even prior to these results, the now widespread use of improved light-quark actions and the availability of ensembles at small lattice spacings made possible lattice-QCD results for *D*-meson decay constants and form factors with uncertainties in the range of $1-5\%$. *B*-meson decay constants have received a lot of attention with a similar (to the *D*-meson case) number of independent lattice-QCD results available, all using different heavy-quark methods. The present situation for *B*-meson form factors and mixing quantities is much more sparse. However, as discussed in Section 5, there are many different processes and quantities to consider and for each process several new results were presented at this conference. Given the level of activity, we should see a significant improvement in the reported uncertainties for many of these quantities in the next few years. This is good news since the *B*-meson decay constants, semileptonic form factors,
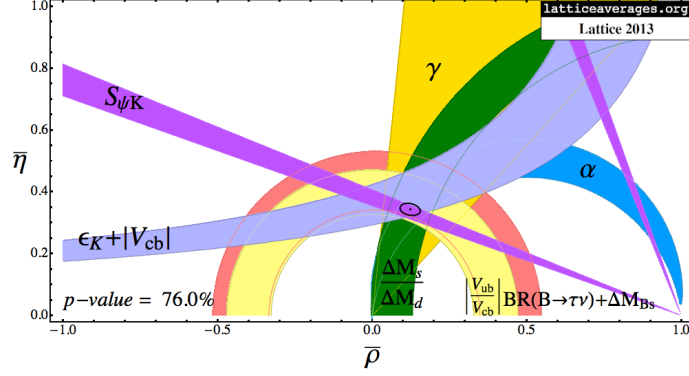
Figure 6: Example of a UT fit obtained in Refs. [5, 118] using the FLAG-2 averages from Ref. [7]. See Refs. [5, 118] for more details.

and mixing parameters are important inputs to CKM fits. Already, lattice calculations of several *D*- and *B*-meson quantities have reached the level of precision where isospin breaking and EM effects are becoming relevant. Strong (and EM induced) isospin breaking effects are straightforward to include, the same as for kaon decays. However, quantitative phenomenological predictions of structure dependent EM effects in (semi)leptonic *D*- and *B*-meson decays similar to the ones for kaon decay [38] (including an error budget), don't currently exist. Such calculations are now needed.

With the FLAG-2 averages in hand, one can now consider how the lattice results affect the SM CKM picture. This is done in a global analysis that takes the relevant averages of experimental measurements together with the FLAG-2 averages for the relevant hadronic parameters as inputs and performs a fit to the SM CKM parameters, in particular, to obtain constraints on ρ and η. Such fits are usually referred to as unitarity triangle fits. There are currently three groups which perform such analyses: LLV [5, 118], the UTfit collaboration [119], and the CKM fitter group [120]. There are some differences in which inputs are used between the three groups, in particular which lattice results are used, and how the errors are treated. Figure 6 shows an example of a unitarity triangle fit obtained from LLV, which I discuss here since it uses the FLAG-2 averages of Ref. [7] for $\hat{B}_K$, f_K, ξ, $f_{B_s}\sqrt{\hat{B}_{B_s}}$, f_B, $|V_{ub}|_{\rm excl}$, and $|V_{cb}|_{\rm excl}$. The exclusive $|V_{u(c)b}|$ are averaged with the inclusive ones using the PDG prescription to inflate the errors when combining inconsistent results. See Refs. [5, 118] for a list of the other inputs. Since the unitarity triangle parameters ρ and η are overconstrained with all the inputs, one can remove constraints and instead obtain predictions for them. The resulting differences provide information in addition to the goodness of fit (*p*-value) of how consistent the various inputs are with each other. If there are inconsistencies, one can then also use this analysis to anticipate where new physics may most likely make an appearance. Unfortunately, however, there currently are no big inconsistencies, unlike a few years ago. Nevertheless, the persistent 3σ differences between inclusive and exclusive $|V_{ub}|$ and $|V_{cb}|$ determinations are still cause for concern.

Acknowledgements

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